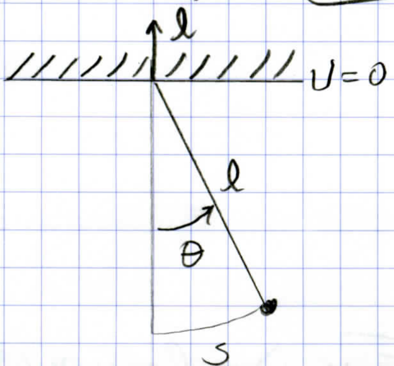
TM5 Pr 7.24

A plane pendulum has its string shortened by

$$\frac{dl}{dt} = \dot{l} = -\alpha = \text{Constant.}$$

Find the Lagrangian & Hamiltonian & the total energy and discuss the conservation of energy in the system.



For constant l:

$$s = l\theta \qquad v = l\dot{\theta}$$

$\vec{v}$ in polar coordinates is $\dot{r} + r\dot{\theta}$. See p. 32 Eq. (1.97)

For changing l

$$s = l\theta \quad \text{but} \quad v = \overbrace{l\dot{\theta} + \dot{l}} \qquad (\text{TM5 } 1.97)$$

Write out energies:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{\theta}+\dot{l})^2 = \frac{1}{2}m(l^2\dot{\theta}^2 + \underbrace{\cancel{2l\dot{\theta}\cdot\dot{l}}}_{0 \text{ since } \dot{\theta}\perp\dot{l}} + \dot{l}^2)$$

$$U = -mgl\cos\theta \qquad (U=0 \text{ at support})$$

The Lagrangian is

$$L = \frac{1}{2}m(l\dot{\theta}^2 + \dot{l}^2) + mgl\cos\theta$$

Substituting $\dot{l} = \alpha$

$$\boxed{L = \frac{1}{2}m(l\dot{\theta}^2 + \alpha^2) + mgl\cos\theta}$$

Find the Hamiltonian: θ is the only coordinate since it expresses the single degree of freedom, l does not express a degree of freedom since it's determined outside the system, hence it's not a coordinate

$$H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$$

$$= \dot{\theta}(ml\dot{\theta}) - \frac{1}{2}m(l\dot{\theta}^2 + \alpha^2) - mgl\cos\theta$$

$$= \frac{1}{2}ml\dot{\theta}^2 - \frac{1}{2}m\alpha^2 - mgl\cos\theta$$

Find p_θ to re-write H

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{ml}, \quad \dot{\theta}^2 = \frac{p_\theta^2}{m^2 l^2}$$

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WRITE THE HAMILTONIAN WITH p_x

$$\boxed{H = \frac{p_\theta^2}{2ml} - \frac{1}{2}m\alpha^2 - mgl\cos\theta}$$

THE TOTAL ENERGY IS

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}m\alpha^2 - mgl\cos\theta$$

OR

$$E = \frac{p_\theta^2}{2ml^2} + \frac{1}{2}m\alpha^2 - mgl\cos\theta$$

$$\boxed{H \neq E!}$$

BECAUSE E IS NOT CONSERVED

$\dot{l}$ IS A SOURCE OF ENERGY